

## Solutions of the Hua System on Hermitian Symmetric Spaces of Tube Type

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### ABSTRACT

In this paper, we give an over view results on the eigen functions of the Hua operator on a Hermitian symmetric space of tube type  $X = G/K$ , let  $\lambda_j \in \mathbb{C}$  ( $j = 1, 2, \dots, n$ ) such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ , and let  $F$  be a  $\mathbb{C}$ -valued function on  $X$  satisfying the following Hua system:

$$H_q F = \sum_{j=1}^n \frac{(\lambda_j^2 - \eta^2)}{32\eta^2} F Z.$$

Then  $F$  has an  $L^p$ -Poisson integral representation ( $1 < p < +\infty$ ) over the Shilov boundary of  $X$ .

**Key words:** Hua operator, tube type, eigenfunctions, Hua system,

### INTRODUCTION

Let  $X = G/K$  be an irreducible bounded symmetric domain of tube type and let  $H_q$  be the associated Hua operator on it. Shimeno [11] proved that the Poisson transform maps the space  $\sum_{j=1}^n B(G/P_{\mathbb{E}}; \lambda_j)$  of hyperfunctions-valued sections of a degenerate principal series attached to the Shilov boundary  $G/P_{\mathbb{E}}$  of  $X$ , bijectively onto an eigenspace  $\sum_{j=1}^n E_{\lambda_j}(X)$  of the Hua operator  $H_q$ , under certain condition on the complex parameter  $\lambda_1, \lambda_2, \dots, \lambda_j$ . Here  $P_{\mathbb{E}}$  is a certain maximal parabolic subgroup of  $G$ . Now let  $r$  and  $m'$  denote respectively the rank of  $X$  and the multiplicity of the short restricted roots.

**Theorem (1):** [11] Let  $\lambda_1, \lambda_2, \dots, \lambda_j$  be a complex number such that

$$\sum_{j=1}^n -\lambda_j - \frac{m'}{2}(-r + 2 + i) \notin \{1, 2, 3, \dots\} \text{ for } i = 0 \text{ and } 1. \quad (1)$$

Then the Poisson transform is a  $G$ -isomorphism from  $\sum_{j=1}^n B(G/P_{\mathbb{E}}; \lambda_j)$  onto

the space  $\sum_{j=1}^n E_{\lambda_j}(X)$  of all analytic functions  $F$  on  $X$  that satisfy

$$H_q F = \sum_{j=1}^n \frac{(\lambda_j^2 - (1 + m'(r-1)/2)^2)}{32(1 + m'(r-1)/2)^2} F Z. \quad (2)$$

$Z$  being some specific element in the center of the Lie algebra of  $K$ .

In the light of the above result, it is natural to look for a characterization of the range of the Poisson transform on classical spaces on the Shilov boundary  $G/P_{\mathbb{E}}$  such as  $C^\infty(G/P_{\mathbb{E}})$ ,  $L^p(G/P_{\mathbb{E}})$  and the space of distribution  $D'(G/P_{\mathbb{E}})$ .

In the case  $\sum_{j=1}^n \lambda_j = 1 + m(r-1)/2$ , i.e. the eigenspace consists of Hua harmonic functions, Koranyi and Malliavin [8] showed in the case of  $X = Sp(2, \mathbb{R})/U(2)$  that the image of the space  $L^\infty(G/P_{\mathbb{E}})$  is the space of all bounded Hua harmonic functions on  $X$ .

Actually, this result remains true for all Hermitian symmetric spaces of tube type, see. [5]

If  $X = SU(n, n)/S(U(n) \times U(n))$ , we showed in [2] that the Poisson transform attached to certain degenerate principal series of  $SU(n, n)$  is a topological isomorphism from  $L^2(G/P_{\mathbb{E}})$  onto a specific Hardy type space for eigenfunctions of the Hua operator on  $X$ .

The aim of this section is, on one hand, to extend in a unified manner the result in [2] to all Hermitian symmetric spaces of tube type and on the other hand, to characterize, for all  $p, 1 < p < +\infty$ , the  $L^p$ -range of the Poisson transform in  $X$ .

Let  $P_{\mathbb{E}}$  be a maximal standard parabolic subgroup of  $G$  with Langlands decomposition  $P_{\mathbb{E}} = M_{\mathbb{E}}A_{\mathbb{E}}N_{\mathbb{E}}$  such that  $A_{\mathbb{E}}$  is of real dimension one. Then, the group  $G$  has the following generalized Iwasawa decomposition:

$$G = KM_{\mathbb{E}}A_{\mathbb{E}}N_{\mathbb{E}}^{\pm}.$$

For  $x \in G$ , we denote by  $H_{\mathbb{E}}(x)$  the unique element in  $a_{\mathbb{E}}$  ( $a_{\mathbb{E}}$  being the Lie algebra of  $A_{\mathbb{E}}$ ), such that  $x \in KM_{\mathbb{E}}e^{H_{\mathbb{E}}(x)}N_{\mathbb{E}}^{\pm}$ . On the one-dimensional Lie algebra  $a_{\mathbb{E}} = \mathbb{R}X_0$  we define the linear forms

$$\rho_0(X_0) = r \text{ and } \rho_{\mathbb{E}} = \left(1 + \frac{m'}{2}(r-1)\right)\rho_0.$$

For  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  we denote by  $\sum_{j=1}^n B(G/P_{\mathbb{E}}; \lambda_j)$  the space of all hyperfunctions  $f$  on  $G$  that satisfy

$$f(gman) = \prod_{j=1}^n e^{(\lambda_j \rho_0 - \rho_{\mathbb{E}})H_{\mathbb{E}}(a)} f(g), \forall g \in G, m \in M_{\mathbb{E}}, a \in A_{\mathbb{E}}, n \in N_{\mathbb{E}}^{\pm}.$$

Then the Poisson transform  $P_{\lambda}$  of an element  $f \in \sum_{j=1}^n B(G/P_{\mathbb{E}}; \lambda_j)$  is defined by

$$\sum_{j=1}^n P_{\lambda_j} f(gK) = \int_K f(gk) dk.$$

A straightforward computation shows that

$$\sum_{j=1}^n P_{\lambda_j} f(gK) = \int_K \prod_{j=1}^n e^{-(\lambda_j \rho_0 + \rho_{\mathbb{E}})H_{\mathbb{E}}(g^{-1}k)} f(k) dk.$$

By the decomposition  $G = KP_{\mathbb{E}}$ , the restriction from  $G$  to  $K_{\mathbb{E}} = K \cap M_{\mathbb{E}}$  gives a  $G$ -isomorphism from  $\sum_{j=1}^n B(G/P_{\mathbb{E}}; \lambda_j)$  onto the space  $B(K/K_{\mathbb{E}})$  of all hyperfunctions  $f$  on  $K$  that satisfy

$$f(km) = f(k), \forall m \in K_{\mathbb{E}}.$$

The classical space  $L^p(K/K_{\mathbb{E}})$  will be regarded as the space of all  $\mathbb{C}$ -valued measurable (classes) functions  $f$  on  $K$  which

are right  $K_{\mathbb{E}}$ -invariant with  $\|f\|_p < +\infty$ . Here

$$\|f\|_p = \left[ \int_K |f(k)|^p dk \right]^{1/p},$$

$dk$  being the normalized Haar measure of the compact group  $K$ . Since the space  $L^p(K/K_{\mathbb{E}})$  can be seen as a  $G$ -invariant subspace of  $\sum_{j=1}^n B(G/P_{\mathbb{E}}; \lambda_j)$ , then the image  $\sum_{j=1}^n P_{\lambda_j}(L^p(K/K_{\mathbb{E}}))$  is a proper closed subspace of  $\sum_{j=1}^n E_{\lambda_j}(X)$ , by Shimeno result, provided that the parameter  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  satisfies (1).

Now, in order to characterize those  $F$  in  $\sum_{j=1}^n E_{\lambda_j}(X)$  that are Poisson transforms by  $\sum_{j=1}^n P_{\lambda_j}$  of some  $f \in L^p(K/K_{\mathbb{E}})$ , we introduce a Hardy type space  $\sum_{j=1}^n E_{\lambda_j, p}^*(X)$  for eigenfunctions of the Hua operator  $H_q$ .

More precisely, for  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  and  $p, 1 < p < +\infty$ , we define  $\sum_{j=1}^n E_{\lambda_j, p}^*(X)$  to be the space of all functions  $F$  in  $\sum_{j=1}^n E_{\lambda_j}(X)$  that satisfy

$$\sum_{j=1}^n \|F\|_{\lambda_j, p} = \sup_{a \in A_{\mathbb{E}}^+} \prod_{j=1}^n e^{(\rho_{\mathbb{E}} - \Re(\lambda_j)\rho_0)H_{\mathbb{E}}(a)} \left( \int_K |F(ka)|^p dk \right)^{1/p} < +\infty.$$

From now on, we will use on  $A_{\mathbb{E}}$  the coordinate  $a_t = e^{tX_0}$ ;  $t \in \mathbb{R}$ . Henceforth the above norm becomes

$$\sum_{j=1}^n \|F\|_{\lambda_j, p} = \sup_{t > 0} \prod_{j=1}^n e^{r(\eta - \Re(\lambda_j))t} \left( \int_K |F(ka_t)|^p dk \right)^{1/p},$$

where  $\eta = 1 + m'(r-1)/2$ .

Notice that  $2\eta$  is the so called genus of the bounded symmetric domain  $X$ .

Also, we introduce a  $c$ -function given by the following integral representation:

$$\sum_{j=1}^n c(\lambda_j) = \int_{N_{\mathbb{E}}} \prod_{j=1}^n e^{-(\lambda_j \rho_0 + \rho_{\mathbb{E}})H_{\mathbb{E}}(n)} dn.$$

The above integral converges absolutely if and only if  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$  (see Lemma (3)).

Notice that the introduced  $c$ -function  $\sum_{j=1}^n c(\lambda_j)$  appears naturally in the study of the intertwining operators associated to the noncompact realization of the degenerate principal series representation of  $G$ . We have the following consequences

(i) As an immediate consequence of Theorem (11), we get that for  $\lambda_1, \lambda_2, \dots, \lambda_j \in$

$\mathbb{C}$  such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$  and for  $p, 1 < p < +\infty$ , the introduced Hardy type spaces  $\sum_{j=1}^n E_{\lambda_j, p}^*(X)$  are Banach spaces. This is closely-similar to O. Bray conjecture in the case of the real hyperbolic space with  $p = 2$  (see [1,12]).

(ii) Letting  $\sum_{j=1}^n \lambda_j = \eta$  in Theorem (11), we get a characterization of those Hua harmonic functions on  $X$  (i.e.  $H_q F = 0$ ) that have an  $L^p$ -Poisson integral representations over the Shilov boundary  $K/K_{\mathbb{E}}$  of  $X$ . More precisely, let  $H^p(X)$  denote the space of all  $\mathbb{C}$ -valued Hua harmonic functions  $F$  on  $X$  such that

$$\sup_{t>0} \left( \int_K |F(ka_t)|^p dk \right)^{1/p} < +\infty.$$

Then we have

**Corollary (2):** For  $1 < p < \infty$ , the Poisson transform  $P_{\eta}$  is a topological isomorphism from  $L^p(K/K_{\mathbb{E}})$  onto  $H^p(X)$ .

(iii) Since holomorphic functions on  $X$  are annihilated by the Hua operator we can use the above corollary to show that every holomorphic function on  $X$  with finite Hardy norm (i.e.  $\sup_{t>0} \left( \int_K |F(ka_t)|^p dk \right)^{1/p} < +\infty$ ) has an  $L^p$ -Poisson integral representation over the Shilov boundary of  $X$ . Such result was earlier established by Koranyi using a different method (see [12,8,6]).

We are concerned on the minimal Hua system  $H_q$  introduced by Lassalle. [9] Since we will not require the general properties of  $H_q$  we will not need to recall its definition referring to Lassalle, [9] Helgason [5] (see also Faraut and Koranyi [3] for Jordan algebra theoretical setting of  $H_q$ ).

In this section we recall some structural results on Hermitian symmetric spaces of tube type from [11] without proof.

For a real Lie algebra  $\mathfrak{b}$  we denote by  $\mathfrak{b}_c$  its complexification. Let  $G$  be a connected simple Lie group with finite center and let  $K$  be a maximal compact subgroup of  $G$ . Suppose that  $G/K$  is a Hermitian symmetric space of tube type and of rank  $r > 1$ .

Let  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$  be the Cartan decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$  with Cartan involution  $\theta$ . The center  $\mathfrak{z}$  of  $\mathfrak{t}$  is of dimension one and there exists  $Z \in \mathfrak{z}$  such that  $(adZ)^2 = -1$  on  $\mathfrak{p}_c$ . Then  $\mathfrak{p}_c$  decomposes as  $\mathfrak{p}^+ \oplus \mathfrak{p}^-$ , the eigenspaces of  $\pm i$ , respectively.

Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{t}$  (hence also of  $\mathfrak{g}$ ). We denote by  $\Delta$  the set of roots of the pair  $(\mathfrak{g}_c, \mathfrak{h}_c)$ . For  $\gamma \in \Delta$ , let  $\mathfrak{g}_{\gamma} \subset \mathfrak{g}_c$  be the root space for  $\gamma$ . We choose a set of positive roots  $\Delta^+$  such that  $\mathfrak{p}^+ = \sum_{\gamma \in \Delta_n^+} \mathfrak{g}_{\gamma}$ , where  $\Delta_n^+ = \Delta^+ \cap \Delta_n$ ,  $\Delta_n$  being the set of noncompact roots.

Let  $B$  denote the Killing form of  $\mathfrak{g}_c$ . For each  $\gamma \in \Delta$  we choose vectors  $H_{\gamma} \in \mathfrak{h}_c, E_{\gamma}$  and  $E_{-\gamma}$  such that  $\overline{E_{\gamma}} = -E_{-\gamma}, [E_{\gamma}, E_{-\gamma}] = H_{\gamma}$ , where the bar denotes the conjugation with respect to the real form  $\mathfrak{t} + i\mathfrak{p}$  of  $\mathfrak{g}_c$ .

For  $\alpha \in \Delta_n^+$ , we set  $X_{\alpha} = E_{\alpha} + E_{-\alpha}$  and  $Y_{\alpha} = i(E_{\alpha} + E_{-\alpha})$ . Then it is well known that  $X_{\alpha}$  and  $Y_{\alpha}$  form a basis of  $\mathfrak{p}$ .

Let  $\{\gamma_1, \dots, \gamma_r\}$  be a maximal set of strongly orthogonal noncompact roots such that  $\gamma_j$  is the highest element of  $\Delta_n^+$  strongly orthogonal to  $\gamma_{j+1}, \dots, \gamma_r$  for  $j = r, \dots, 1$ . Then the space  $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}X_{\gamma_j}$  is a maximal Abelian subspace of  $\mathfrak{p}$ .

Let  $\Sigma$  (respectively  $\Sigma^+$ ) denote the set of restricted roots of the pair  $(\mathfrak{g}, \mathfrak{a})$  (respectively positive roots in  $\Sigma$ ). Then by classification,  $\Sigma^+$  is of the form

$$\Sigma^+ = \left\{ \beta_i, \frac{\beta_j \pm \beta_k}{2}, 1 \leq i \leq r, 1 \leq k < j \leq r \right\},$$

where  $\beta_i = \gamma_i \circ (c^{-1})|_{\mathfrak{a}}$ ,  $c$  being the Cayley transform of  $\mathfrak{g}_c$ .

We set

$$\alpha_j = \frac{\beta_{r-j+1} - \beta_{r-j}}{2}$$

for  $1 \leq j \leq r - 1$  and  $\alpha_r = \beta_1$ .

Then  $\Gamma = \{\alpha_1, \dots, \alpha_r\}$ , is the set of simple roots in  $\Sigma^+$ .

For  $\alpha \in \Sigma$  let  $\mathfrak{g}^{\alpha} \subset \mathfrak{g}$  be the root space for  $\alpha$  and  $m_{\alpha}$  its multiplicity. As usual put  $\rho = (1/2) \sum_{\alpha \in \Sigma^+} m_{\alpha} \alpha$ . The multiplicity of long roots  $m_{\beta_i}$  for  $i = 1, 2, \dots, r$  equals 1. We set  $m' = m_{1/2 \pm \beta_j \pm \beta_k}$  for  $1 \leq j \neq k \leq r$ .

For  $\sum_{j=1}^n \lambda_j \in a_c^*$ , we denote by  $\sum_{j=1}^n H_{\lambda_j}$  the unique element in  $a_c$  such that  $\sum_{j=1}^n B(H, H_{\lambda_j}) = \sum_{j=1}^n H_{\lambda_j}$  for all  $H \in a$ .

For  $\lambda_1, \lambda_2, \dots, \lambda_j, \mu \in a_c^*$  we put  $\sum_{j=1}^n \langle \lambda_j, \mu \rangle = \sum_{j=1}^n B(H_{\lambda_j}, H_\mu)$ . Let  $W$  be the Weyl group of the pair  $(g, a)$ . Then  $W$  acts on  $a$  and  $a^*$  (via the Killing form) and is naturally identified with the Weyl group of  $\Sigma$ .

Let  $\mathcal{E} = \Gamma \setminus \{\alpha_r\}$  and let  $P_{\mathcal{E}}$  be the corresponding standard parabolic subgroup with the Langlands decomposition  $P_{\mathcal{E}} = M_{\mathcal{E}} A_{\mathcal{E}} N_{\mathcal{E}}^+$  such that  $A_{\mathcal{E}} \subset A$ , where  $A$  is the analytic subgroup of  $G$  with Lie algebra  $a$ . Then, it is well known that  $P_{\mathcal{E}}$  is a maximal standard parabolic subgroup of  $G$  and  $G/P_{\mathcal{E}}$  is the Shilov boundary of  $X$ . Moreover,  $G/P_{\mathcal{E}}$  can be identified to the compact symmetric space  $K/K_{\mathcal{E}}$ . If  $a_{\mathcal{E}}$  denotes the Lie algebra of  $A_{\mathcal{E}}$ . Then

$$a_{\mathcal{E}} = \{H \in a; \gamma(h) = 0, \forall \gamma \in \mathcal{E}\}.$$

Moreover,  $a_{\mathcal{E}} = \mathbb{R}X_0$ , where  $X_0 = \sum_{j=1}^r X_{\gamma_j}$ .

Let  $a(\mathcal{E})$  denote the orthogonal complement of  $a_{\mathcal{E}}$  in  $a$  with respect to the Killing form of  $g$ . Let  $\rho_{\mathcal{E}}$  and  $\rho_{a(\mathcal{E})}$  be the restrictions of  $\rho$  to  $a_{\mathcal{E}}$  and  $a(\mathcal{E})$ , respectively. Then  $\rho = \rho_{\mathcal{E}} + \rho_{a(\mathcal{E})}$ . Moreover,

$$\rho_{\mathcal{E}} = \frac{1}{2} \sum_{\alpha \in \Sigma^+ \setminus \langle \mathcal{E} \rangle} m_{\alpha} \alpha,$$

and

$$\rho_{a(\mathcal{E})} = \frac{1}{2} \sum_{\alpha \in \Sigma^+ \cap \langle \mathcal{E} \rangle} m_{\alpha} \alpha.$$

Here  $\langle \mathcal{E} \rangle = \Sigma \cap \sum_{\alpha \in \mathcal{E}} \mathbb{Z}\alpha$ .

Finally, we recall an integral formula on the group  $N_{\mathcal{E}}^- = \theta(N_{\mathcal{E}}^+)$ . Let  $dn$  be the invariant measure on  $N_{\mathcal{E}}^-$ , normalized by  $\int_{N_{\mathcal{E}}^-} e^{-2\rho_{\mathcal{E}} H_{\mathcal{E}}(n)} dn = 1$ . Then, for a continuous function  $f$  on  $K/K_{\mathcal{E}}$  we have

$$\int_K f(k) dk = \int_{N_{\mathcal{E}}^-} f(\kappa(n)) e^{-2\rho_{\mathcal{E}} H_{\mathcal{E}}(n)} dn. \quad (3)$$

In the above formula  $\kappa(x)$  denotes the  $K$ -component of  $x \in G$  with respect to the decomposition  $G = KM_{\mathcal{E}} A_{\mathcal{E}} N_{\mathcal{E}}^-$ .

We end by a result on representation of compact group which will be useful in

the sequel see. [6] Let  $\widehat{K}$  be the set of all equivalence (classes) finitedimensional irreducible representations of the compact group  $K$ . For  $\delta \in \widehat{K}$ , let  $\mathcal{C}(K/K_{\mathcal{E}})(\delta)$  be the linear span of all  $K$ -finite functions on  $K/K_{\mathcal{E}}$  of type  $\delta$ . Then, the algebraic sum  $\bigoplus_{\delta \in \widehat{K}} \mathcal{C}(K/K_{\mathcal{E}})(\delta)$  is dense in  $\mathcal{C}(K/K_{\mathcal{E}})$  under the topology of uniform convergence. Therefore  $\bigoplus_{\delta \in \widehat{K}} \mathcal{C}(K/K_{\mathcal{E}})(\delta)$  is dense in  $L^p(K/K_{\mathcal{E}})$ .

Before giving the proof of our theorem of Fatou-type stated, we first show that the integral defining the  $c$ -function  $\sum_{j=1}^n c(\lambda_j)$  is absolutely convergent if  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ .

For this, we recall a result on the partial Harish-Chandra  $c^{\mathcal{E}}$ -function (see [10]). Let  $\mu \in a_c^*$ . Then, the integral representation of  $c^{\mathcal{E}}$  is given by

$$c^{\mathcal{E}}(\mu) = \int_{N_{\mathcal{E}}^-} e^{-(\mu+\rho)(H(n))} dn,$$

where  $H(n) \in a$  with respect to the Iwasawa decomposition  $G = KAN$  of  $G$ ,  $x = \kappa(x)e^{H(x)}n(x)$  for  $x \in G$ . The above integral converges absolutely (see [10]) if

$$\Re(\langle \mu, \alpha \rangle) > 0, \forall \alpha \in \Sigma^+ \setminus \langle \mathcal{E} \rangle.$$

**Lemma (3):** [13] Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . Then the integral

$$\sum_{j=1}^n c(\lambda_j) = \int_{N_{\mathcal{E}}^-} \prod_{j=1}^n e^{-(\lambda_j \rho_0 + \rho_{\mathcal{E}}) H_{\mathcal{E}}(n)} dn.$$

converges absolutely.

Proof. Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . Define the following  $\mathbb{C}$ -linear form  $\sum_{j=1}^n \mu_{\lambda_j}$  on  $a_c$ :

$$\sum_{j=1}^n \mu_{\lambda_j}(H) = \sum_{j=1}^n (\lambda_j \rho_0 - \rho_{\mathcal{E}})(H_{\mathcal{E}}) + \rho(H),$$

where  $H_{\mathcal{E}}$  is the  $a_{\mathcal{E}}$ -component of  $H$  with respect to the orthogonal decomposition  $a = a_{\mathcal{E}} \oplus a(\mathcal{E})$ . Notice that the condition  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$  is equivalent to  $\sum_{j=1}^n \langle \mu_{\lambda_j}, \alpha \rangle > 0$  for all  $\alpha \in \Sigma^+$ .

Next let  $w \in \{w \in W; wH = H, \forall H \in a_{\mathcal{E}}\}$  such that

- (i)  $w(\Sigma^+ \cap \langle \mathcal{E} \rangle) = -\Sigma^+ \cap \langle \mathcal{E} \rangle$  and
  - (ii)  $w(\Sigma^+ \setminus \langle \mathcal{E} \rangle) = \Sigma^+ \setminus \langle \mathcal{E} \rangle$ .
- It is easy to see that  $\sum_{j=1}^n w\mu_{\lambda_j} + \rho = \sum_{j=1}^n \lambda_j \rho_0 + \rho_{\mathcal{E}}$ . Since  $\sum_{j=1}^n \langle w\mu_{\lambda_j}, \alpha \rangle = \sum_{j=1}^n \langle \mu_{\lambda_j}, w^{-1}\alpha \rangle$  we get
- $\sum_{j=1}^n \Re(\langle w\mu_{\lambda_j}, \alpha \rangle) > 0, \forall \alpha \in \Sigma^+ \setminus \langle \mathcal{E} \rangle$  by (i) and  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ .

Hence the integral

$$\sum_{j=1}^n c^{\mathcal{E}}(w\mu_{\lambda_j}) = \int_{N_{\mathcal{E}}^-} \prod_{j=1}^n e^{-(w\mu_{\lambda_j} + \rho)(H(n))} dn,$$

converges absolutely and since the above integral is nothing but  $\sum_{j=1}^n c(\lambda_j)$ , the result follows.

For the proof of Theorem (4), we will need the following cocycle relations for the generalized Iwasawa function  $H_{\mathcal{E}}(x)$ :

$$H_{\mathcal{E}}(x\kappa(y)) = H_{\mathcal{E}}(xy) - H_{\mathcal{E}}(y) \tag{4}$$

for all  $x, y \in G$ , and

$$H_{\mathcal{E}}(na^{-1}) = H_{\mathcal{E}}(n) - H_{\mathcal{E}}(a) \tag{5}$$

for  $n \in N_{\mathcal{E}}^-$  and  $a \in A_{\mathcal{E}}$ .

$$\sum_{j=1}^n P_{\lambda_j} f(ka_t) = \int_{N_{\mathcal{E}}^-} \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0(H_{\mathcal{E}}(a_{-t}h))} e^{-2\eta\rho_0(H_{\mathcal{E}}(n))} f(k\kappa(n)) dn.$$

Next use (4) to get

$$\sum_{j=1}^n P_{\lambda_j} f(ka_t) = \int_{N_{\mathcal{E}}^-} \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0(H_{\mathcal{E}}(a_{-t}h))} e^{(\lambda_j - \eta)\rho_0(H_{\mathcal{E}}(n))} f(k\kappa(n)) dn.$$

Now, using on one hand the change of the variables  $n \rightarrow a_{-t}na_t$  and on the other hand the identity (5), the above integral becomes

$$\prod_{j=1}^n e^{(\lambda_j - \eta)rt} \int_{N_{\mathcal{E}}^-} \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0 H_{\mathcal{E}}(n) + (\lambda_j - \eta)\rho_0 H_{\mathcal{E}}(a_t n a_{-t})} f(k\kappa(a_t n a_{-t})) dn.$$

Since  $a_t n a_{-t} \rightarrow e$ , as  $t$  goes to  $+\infty$ , we deduce that

$$\lim_{t \rightarrow +\infty} \prod_{j=1}^n e^{(\eta - \lambda_j)rt} (P_{\lambda_j} f)(ka_t) = \sum_{j=1}^n c(\lambda_j) f(k),$$

provided we justify the reversal order of the limit and integration.

For this, let

$$\psi_t(n) = \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0 H_{\mathcal{E}}(n) + (\lambda_j - \eta)\rho_0 H_{\mathcal{E}}(a_t n a_{-t})} f(k\kappa(a_t n a_{-t})).$$

Putting  $M = \sup_{k \in K} |f(k)|$ , we have

$$|\psi_t(n)| \leq \prod_{j=1}^n e^{-(\Re \lambda_j + \eta)\rho_0 H_{\mathcal{E}}(n) + (\Re \lambda_j - \eta)\rho_0 H_{\mathcal{E}}(a_t n a_{-t})} M.$$

To end the proof we will need the following result which is a particular case of, [10]

**Lemma (5):** Let  $t > 0$  and let  $n \in N_{\mathcal{E}}^-$ . Then we have

$$0 \leq \rho_0(H_{\mathcal{E}}(a_t n a_{-t})) \leq \rho_0(H_{\mathcal{E}}(n)).$$

**Theorem(4):** Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . Then we have:

$$f(k) = \sum_{j=1}^n c(\lambda_j)^{-1} \lim_{t \rightarrow +\infty} \prod_{j=1}^n e^{r(\eta - \lambda_j)t} P_{\lambda_j} f(ka_t),$$

- (i) uniformly for  $f \in C(K/K_{\mathcal{E}})$ ;
- (ii) in  $L^p(K/K_{\mathcal{E}})$ , if  $f \in L^p(K/K_{\mathcal{E}})$ ,  $1 < p < +\infty$ .

Proof. (i) Let  $f$  in  $C(K/K_{\mathcal{E}})$ . Write  $\sum_{j=1}^n P_{\lambda_j} f(ka_t)$  as

$$\sum_{j=1}^n P_{\lambda_j} f(ka_t) = \int_K \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0(H_{\mathcal{E}}(a_{-t}h))} f(kh) dh.$$

Since the integrand

$$h \rightarrow \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0(H_{\mathcal{E}}(a_{-t}h))} f(kh)$$

is a  $K_{\mathcal{E}}$ -invariant function in  $K$ , we can use the integral formula (3) to transform the above integral into an integral over  $N_{\mathcal{E}}^-$ :

Using the above lemma, it is easy to see that  $\psi_t$  is dominated by the function

$$\begin{cases} \prod_{j=1}^n e^{-(\Re(\lambda_j)+\eta)\rho_0 H_{\Xi}}(n), & \text{if } -1 < \Re(\lambda_j) - \eta \leq 0, \\ e^{-2\eta\rho_0 H_{\Xi}}(n), & \text{if } \Re(\lambda_j) - \eta > 0 \end{cases}$$

which is integrable and the result follows.

For the proof of the  $L^p$ -counterpart of Theorem (4), we will need the following result.

Lemma (6): Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  [13] such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . Then, there exists a positive constant  $\sum_{j=1}^n \gamma(\lambda_j)$  such that for  $p > 1$  and  $f \in L^p(K/K_{\Xi})$ , we have:

$$\left[ \int_K \sum_{j=1}^n |P_{\lambda_j} f(ka_t)|^p dk \right]^{1/p} \leq \prod_{j=1}^n \gamma(\lambda_j) e^{(\Re(\lambda_j)-\eta)rt} \|f\|_p. \quad (6)$$

Proof. For fixed  $t > 0$ , define the function  $\sum_{j=1}^n P_{\lambda_j}^t$  on  $K$  by

$$\sum_{j=1}^n P_{\lambda_j}^t(k) = \prod_{j=1}^n e^{-(\lambda_j+\eta)\rho_0 H_{\Xi}(a_{-t}k^{-1})}.$$

Then the Poisson integral  $\sum_{j=1}^n P_{\lambda_j} f$  can be written as a convolution over the compact group  $K$ :

$$\sum_{j=1}^n P_{\lambda_j} f(ka_t) = \sum_{j=1}^n f \star P_{\lambda_j}^t(k).$$

Hence using the Hausdorff–Young inequality, we obtain

$$\left[ \int_K \sum_{j=1}^n |P_{\lambda_j} f(ka_t)|^p dk \right]^{1/p} \leq \sum_{j=1}^n \|f\|_p \|P_{\lambda_j}^t\|_1.$$

Next, since

$$\sum_{j=1}^n \|P_{\lambda_j}^t\|_1 = \int_K \prod_{j=1}^n e^{-(\Re(\lambda_j)+\eta)\rho_0 H_{\Xi}(a_{-t}k)} dk,$$

$$\begin{aligned} \prod_{j=1}^n \|c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda_j}^t(f) - f\|_p &\leq \prod_{j=1}^n \|c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda_j}^t(f - \phi)\|_p \\ &\quad + \prod_{j=1}^n \|c(\lambda_j)^{-1} e^{(\eta-\lambda_j)rt} P_{\lambda_j}^t \phi - \phi\|_p \end{aligned}$$

$$+ \| \phi - f \|_p, \quad (7)$$

where  $\sum_{j=1}^n P_{\lambda_j}^t f(k) = \sum_{j=1}^n P_{\lambda_j} f(ka_t)$ .

The first term in the right-hand side of (7) is less than

$$\sum_{j=1}^n \gamma(\lambda_j) |c(\lambda_j)|^{-1} \| \phi - f \|_p,$$

(i.e.  $\sum_{j=1}^n \|P_{\lambda_j}^t\|_1 = \sum_{j=1}^n P_{\Re(\lambda_j)} 1(a_t)$ ), we deduce from the part one of Theorem (4) that there exists a positive constant  $\sum_{j=1}^n \gamma(\lambda_j)$  such that

$$\sum_{j=1}^n \|P_{\lambda_j}^t\|_1 \leq \prod_{j=1}^n \gamma(\lambda_j) e^{r(\Re(\lambda_j)-\eta)t},$$

which implies that

$$\sup_{t>0} \prod_{j=1}^n e^{r(\eta-\Re(\lambda_j))t} \left[ \int_K \sum_{j=1}^n |P_{\lambda_j} f(ka_t)|^p dk \right]^{1/p} \leq \sum_{j=1}^n \gamma(\lambda_j) \|f\|_p,$$

and the proof of Lemma (6) is finished.

Now, we give the proof of (ii) of Theorem (4). Let  $f \in L^p(K/K_{\Xi})$ . Then, for any  $\epsilon > 0$ , there exists  $\phi \in \bigoplus_{\delta \in \mathbb{R}} C(K/K_{\Xi})(\delta)$  such that  $\|f - \phi\|_p \leq \epsilon$ . We have

by Lemma (6).

Since  $\phi$  is continuous on  $K/K_{\Xi}$  the (i) part of Theorem (4) shows that

$$\lim_{t \rightarrow +\infty} \prod_{j=1}^n \|c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda_j}^t \phi - \phi\|_p = 0.$$

Therefore,

$$\lim_{t \rightarrow +\infty} \prod_{j=1}^n \|c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda_j} f - f\|_p \leq \sum_{j=1}^n \epsilon(\gamma(\lambda_j) + 1),$$

which implies (ii) and the proof of Theorem (4) is completed.

As a consequence of the  $L^p$ -Fatou-type theorem we get the following estimates on the Poisson transform on  $L^p(K/K_{\mathbb{E}})$ .

**Corollary (7):** Let  $\lambda_1, \lambda_2, \dots, \lambda_j$  [13] be a complex number such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . There exists a positive constant  $\sum_{j=1}^n \gamma(\lambda_j)$  such that for  $1 < p < +\infty$  and  $f \in L^p(K/K_{\mathbb{E}})$ , we have

$$\sum_{j=1}^n |c(\lambda_j)| \|f\|_p \leq \sum_{j=1}^n \|P_{\lambda_j} f\|_{\lambda_j, p} \leq \sum_{j=1}^n \gamma(\lambda_j) \|f\|_p.$$

$$\|f\|_p \leq \prod_{j=1}^n |c(\lambda_j)|^{-1} e^{r(\eta-\Re(\lambda_j))t_j} \sup_j \left[ \int_K \sum_{j=1}^n |P_{\lambda_j} f(ka_{t_j})|^p dk \right]^{1/p},$$

which gives  $\|f\|_p \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|P_{\lambda_j} f\|_{\lambda_j, p}$  and the proof of Corollary (3.3.7) is finished.

Recall that the group  $K$  acts on  $L^2(K/K_{\mathbb{E}})$  by  $\pi(h)f(k) = f(h^{-1}k)$  and under this action, the space  $L^2(K/K_{\mathbb{E}})$  has the following Peter-Weyl decomposition  $L^2(K/K_{\mathbb{E}}) = \bigoplus_{\delta \in \widehat{K_0}} V_{\delta}$ , where  $\widehat{K_0}$  denotes the set of all class one (with respect to  $K_{\mathbb{E}}$ ) equivalence-classes-irreducible representations of  $K$  and  $V_{\delta}$  is the finite linear span of  $\{\phi_{\delta} \circ k; k \in K\}$ ,  $\phi_{\delta}$  being the zonal spherical function.

**Proposition (8):** Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  and let  $f \in V_{\delta}$ . Then

$$\sum_{j=1}^n P_{\lambda_j} f(ka_t) = \sum_{j=1}^n \Phi_{\lambda_j, \delta}(a_t) f(k),$$

where  $\sum_{j=1}^n \Phi_{\lambda_j, \delta}(a_t) = \sum_{j=1}^n P_{\lambda_j} \phi_{\delta}(a_t)$ .

Proof. For each  $t \in \mathbb{R}$ , define the operator  $\sum_{j=1}^n P_{\lambda_j}^t$  on  $L^2(K/K_{\mathbb{E}})$  by  $\sum_{j=1}^n P_{\lambda_j}^t f(k) = \sum_{j=1}^n P_{\lambda_j} f(ka_t)$ . Since  $M_{\mathbb{E}}$  centralizes  $A_{\mathbb{E}}$ ,  $\sum_{j=1}^n P_{\lambda_j}^t$  defines a

Proof. We have only to prove the left-hand side of the above estimates. Let  $f \in L^p(K/K_{\mathbb{E}})$ . By (ii) of the previous theorem we know that

$$f(k) = \lim_{t \rightarrow +\infty} \prod_{j=1}^n c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda_j} f(ka_t),$$

in  $L^p(K/K_{\mathbb{E}})$ . Hence, there exists a sequence  $(t_j)$  with  $t_j \rightarrow +\infty$  as  $j \rightarrow +\infty$  such that

$$f(k) = \lim_{j \rightarrow \infty} \prod_{j=1}^n c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t_j} P_{\lambda_j} f(ka_{t_j}),$$

almost every where in  $K$ . By the classical Fatou lemma, we have

bounded operator in  $L^2(K/K_{\mathbb{E}})$ . Also, we can see easily that  $\sum_{j=1}^n P_{\lambda_j}^t$  commutes with  $\pi$ . Hence, by Schur lemma

$$\sum_{j=1}^n P_{\lambda_j}^t = \sum_{j=1}^n \Phi_{\lambda_j, \delta}(a_t) I$$

on each  $V_{\delta}$ , with  $\sum_{j=1}^n \Phi_{\lambda_j, \delta}(a_t) = \sum_{j=1}^n P_{\lambda_j}^t \phi_{\delta}(e)$ .

From Theorem (4) we deduce the following corollary given the asymptotic behaviour of the generalized spherical function  $\sum_{j=1}^n \Phi_{\lambda_j, \delta}$ .

**Corollary (9):** Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . Then

$$\lim_{t \rightarrow +\infty} e^{r(\eta-\lambda)t} \Phi_{\lambda, \delta}(a_t) = c(\lambda)$$

for each  $\delta \in \widehat{K_0}$ .

**Theorem (10):** Let  $\lambda$  be a complex number such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ . Then we have:

(i) A  $\mathbb{C}$ -valued function  $F$  on  $X$  satisfying the Hua system (2) is the Poisson transform by  $P_{\lambda}$  of some  $f \in L^2(K/K_{\mathbb{E}})$  if and only if it satisfies  $\sum_{j=1}^n \|F\|_{\lambda_j, 2} < +\infty$ .

Moreover, there exists a positive constant  $\sum_{j=1}^n \gamma(\lambda_j)$  such that for  $f \in L^2(K/K_{\mathbb{E}})$  the following estimates hold:

$$\sum_{j=1}^n |c(\lambda_j)| \|f\|_2 \leq \sum_{j=1}^n \|P_{\lambda_j} f\|_{\lambda_j, 2} \leq \sum_{j=1}^n \gamma(\lambda_j) \|f\|_2. \quad (8)$$

$$f(k) = \sum_{j=1}^n |c(\lambda_j)|^{-2} \lim_{t \rightarrow +\infty} \prod_{j=1}^n e^{2r(\eta - \Re(\lambda_j))t} \int_K \overline{e^{-(\lambda_j \rho_0 + \rho_{\mathbb{E}})H_{\mathbb{E}}(a_{-t} k^{-1} h)}} F(ha) dh,$$

in  $L^2(K/K_{\mathbb{E}})$ .

**Proof.**

(i) The necessary condition follows from Lemma (6), for  $p = 2$ .

To prove the sufficiency condition, let  $F \in \sum_{j=1}^n E_{\lambda_j, 2}^*(X)$ . Since  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ , we have  $F = \sum_{j=1}^n P_{\lambda_j} f$  for some  $f \in B(K/K_{\mathbb{E}})$ , by Shimeno result.

Let  $f = \sum_{\delta \in \widehat{K}_0} f_{\delta}$  be its  $K$ -type series. Then, using Proposition (8),  $F$  can be written as

$$F(ka_t) = \sum_{\delta \in \widehat{K}_0} \sum_{j=1}^n \Phi_{\lambda_j, \delta}(a_t) f_{\delta}(k)$$

in  $C^{\infty}(K \times [0, +\infty))$ .

Next, since  $\sum_{j=1}^n \|F\|_{\lambda_j, 2} < +\infty$ , we have

$$\prod_{j=1}^n e^{2r(\eta - \Re(\lambda_j))t} \sum_{\delta \in \widehat{K}_0} \sum_{j=1}^n |\Phi_{\lambda_j, \delta}(a_t)|^2 \|f_{\delta}\|_2^2 < +\infty$$

for every  $t > 0$ .

Let  $\Omega$  be a finite subset of  $\widehat{K}_0$ . Then, using the asymptotic behaviour of  $\sum_{j=1}^n \Phi_{\lambda_j, \delta}$  given by Corollary (9), we see that

$$\sum_{j=1}^n |c(\lambda_j)|^2 \sum_{\delta \in \Omega} \|f_{\delta}\|_2^2 \leq \sum_{j=1}^n \|F\|_{\lambda_j, 2}^2.$$

in  $C^{\infty}(K)$ .

Now from

$$\|g_t - f\|_2^2 = \sum_{\delta \in \widehat{K}_0} \prod_{j=1}^n \left| |c(\lambda_j)|^{-2} e^{2(\eta - \Re(\lambda_j))rt} |\Phi_{\lambda_j, \delta}(a_t)|^2 - 1 \right|^2 \|f_{\delta}\|_2^2,$$

and  $\lim_{t \rightarrow +\infty} \prod_{j=1}^n e^{(\eta - \Re(\lambda_j))rt} \Phi_{\lambda_j, \delta} = \sum_{j=1}^n c(\lambda_j)$ , we deduce that  $\lim_{t \rightarrow +\infty} \|g_t - f\|_2 = 0$ , and the proof of Theorem (10) is completed.

**Theorem (11):** Let  $\lambda_1, \lambda_2, \dots, \lambda_j \in \mathbb{C}$  such that  $\sum_{j=1}^n \Re(\lambda_j) > \eta - 1$ , and let  $p, 1 < p < +\infty$ . Then we have a function  $F \in$

(ii) Let  $F \in \sum_{j=1}^n E_{\lambda_j, 2}^*(X)$ . Then its  $L^2$ -boundary value  $f$  is given by the following inversion formula:

Since  $\Omega$  is arbitrary, it follows that  $f = \sum_{\delta \in \widehat{K}_0} f_{\delta} \in L^2(K/K_{\mathbb{E}})$  and that  $\sum_{j=1}^n |c(\lambda_j)| \|f\|_2 \leq \sum_{j=1}^n \|P_{\lambda_j} f\|_{*, 2}$ . This finishes the proof of the first part of Theorem (10).

(ii) Now, we turn to the proof of the  $L^2$ -inversion formula.

Let  $F \in \sum_{j=1}^n E_{\lambda_j, 2}^*(X)$ . By the first part of Theorem (10), we know that there exists a unique  $f \in L^2(K/K_{\mathbb{E}})$  such that  $F = \sum_{j=1}^n P_{\lambda_j} f$ . Hence, expanding  $f$  into its  $K$ -type series,  $f = \sum_{\delta \in \widehat{K}_0} f_{\delta}$ , Proposition (8) shows that

$$F(ka_t) = \sum_{\delta \in \widehat{K}_0} \sum_{j=1}^n \Phi_{\lambda_j, \delta}(a_t) f_{\delta}(k)$$

in  $C^{\infty}(K \times [0, +\infty])$ .

Next, for each  $t > 0$  we define a  $\mathbb{C}$ -valued function  $g_t$  on  $K$  by

$$g_t(k) = |c(\lambda)|^{-2} e^{2r(\eta - \Re(\lambda))t} \int_K \overline{e^{-(\lambda + \eta)\rho_0 H_{\mathbb{E}}(a_{-t} k^{-1} h)}} F(ha) dh.$$

Then, replacing  $F$  by its series expansion and using again Proposition (8), we see that  $g_t$  can be rewritten as

$$g_t(k) = \prod_{j=1}^n |c(\lambda_j)|^{-2} e^{2(\eta - \Re(\lambda_j))rt} \sum_{\delta \in \widehat{K}_0} \sum_{j=1}^n |\Phi_{\lambda_j, \delta}(a_t)|^2 f_{\delta}(k)$$

$\sum_{j=1}^n E_{\lambda_j}(X)$  is the Poisson transform by  $\sum_{j=1}^n P_{\lambda_j}$  of some  $f \in L^p(K/K_{\mathbb{E}})$  if and only if  $F \in \sum_{j=1}^n E_{\lambda_j, p}^*(X)$ .

Moreover, there exists a positive constant  $\sum_{j=1}^n \gamma(\lambda_j)$  such that for  $f \in L^p(K/K_{\mathbb{E}})$  the following estimates hold:

$$\sum_{j=1}^n |c(\lambda_j)| \|f\|_p \leq \sum_{j=1}^n \|P_{\lambda_j} f\|_{\lambda_j, p} \leq \sum_{j=1}^n r(\lambda_j) \|f\|_p. \quad (9)$$

Proof.

The “if” part follows from Lemma (6). The proof of the converse will be divided into two parts.

$$g_t(k) = \prod_{j=1}^n |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \int_K \prod_{j=1}^n \overline{e^{-(\lambda_j + \eta)\rho_0 H_{\Xi}(a_{-t} k^{-1} h)}} F(h a_t) dh.$$

Let  $\phi$  be a continuous function in  $K/K_{\Xi}$ . Then we have

$$\lim_{t \rightarrow +\infty} \int_K g_t(k) \overline{\phi(k)} dk = \int_K f(k) \overline{\phi(k)} dk.$$

But

$$\int_K g_t(k) \overline{\phi(k)} dk = \prod_{j=1}^n |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \int_K \left[ \int_K \prod_{j=1}^n \overline{e^{-(\lambda_j + \eta)\rho_0 H_{\Xi}(a_{-t} h^{-1} k)}} F(h a_t) dh \right] \overline{\phi(k)} dk.$$

Observing that

$$H_{\Xi}(a_t k) = H_{\Xi}(a_t k^{-1}),$$

for every  $k \in K$  and using Fubini theorem, we can rewrite the right-hand side of the above equality as

$$\prod_{j=1}^n |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \int_K \prod_{j=1}^n \overline{P_{\lambda_j} \phi(h a_t)} F(h a_t) dh,$$

which is—by the Hölder inequality—majorized by

$$\prod_{j=1}^n |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \left[ \int_K \sum_{j=1}^n |P_{\lambda_j} \phi(h a_t)|^q dh \right]^{1/q} \left[ \int_K |F(h a_t)|^p dh \right]^{1/p},$$

where  $q$  is such that  $1/q + 1/p = 1$ .

Since  $F \in \sum_{j=1}^n E_{\lambda_j, p}^*(X)$ , we obtain

$$\left| \int_K g_t(k) \overline{\phi(k)} dk \right| \leq \prod_{j=1}^n |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \sum_{j=1}^n \left[ \int_K |P_{\lambda_j} \phi(h a_t)|^q dh \right]^{1/q} \|F\|_{\lambda_j, p}.$$

By Theorem (10) we know that

$$\phi(k) = \sum_{j=1}^n c(\lambda_j)^{-1} \lim_{t \rightarrow +\infty} \prod_{j=1}^n e^{r(\eta - \lambda_j)t} P_{\lambda_j} \phi(k a_t)$$

in  $L^q(K/K_{\Xi})$ . Hence

$$\left| \int_K f(k) \overline{\phi(k)} dk \right| \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|\phi\|_q \|F\|_{\lambda_j, p}.$$

Finally, taking the supremum over all continuous  $\phi$  with  $\|\phi\|_q = 1$  in the above inequality we deduce that  $f \in L^p(K/K_{\Xi})$  and that  $\sum_{j=1}^n |c(\lambda_j)| \|f\|_p \leq \sum_{j=1}^n \|F\|_{\lambda_j, p}$ , which is the desired result.

(i) The case  $p \geq 2$ . Firstly, observe that in this case  $\sum_{j=1}^n E_{\lambda_j, p}^*(X) \subset \sum_{j=1}^n E_{\lambda_j, 2}^*(X)$ . Hence, for a given  $F \in \sum_{j=1}^n E_{\lambda_j, p}^*(X)$ , we know by Theorem (10) that there exists  $f \in L^2(K/K_{\Xi})$  such that  $F = \sum_{j=1}^n P_{\lambda_j} f$  and that the function  $f$  can be recovered from  $F$  via the  $L^2$ -type inversion formula  $f(k) = \lim_{t \rightarrow +\infty} g_t(k)$  in  $L^2(K)$ , where

$$\int_K g_t(k) \overline{\phi(k)} dk = \int_K f(k) \overline{\phi(k)} dk.$$

(ii) Part 2. The case  $1 < p \leq 2$ . Let  $\chi_n$  be an approximation of the identity in  $C(K)$ . That is,

$$\chi_n \geq 0, \quad \int_K \chi_n(k) dk = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \int_{K/V} \chi_n(k) dk = 0$$

for every neighborhood  $V$  of  $e$  in  $K$ .

Put

$$F_n(g) = \int_K \chi_n(k) F(k^{-1} g) dk.$$

Then,  $\lim_{n \rightarrow +\infty} F_n = F$  point wise in  $G$ . Since the eigenspace  $\sum_{j=1}^n E_{\lambda_j}(X)$  is  $G$ -invariant,  $F_n$  lies also in  $\sum_{j=1}^n E_{\lambda_j}(X)$ . For each  $t > 0$  define a function  $F_n^t$  in  $K$  by

$F_n^t(k) = F_n(ka_t)$ . Then  $F_n^t = \chi_n \star F^t$ .  
 Moreover, we have

$$\|F_n^t\|_2 \leq \|\chi_n\|_2 \|F^t\|_1 \leq \|\chi_n\|_2 \|F^t\|_p.$$

From the above inequalities we see that for each  $n$  the defined functions  $F_n$  lies in the space  $\sum_{j=1}^n E_{\lambda_j, 2}^*(X)$ . Hence, there exists  $f_n \in L^2(K/K_{\mathbb{E}})$  such that  $F_n = \sum_{j=1}^n P_{\lambda_j} f_n$ , by Theorem (10).

Let  $q$  be a positive number such that  $1/p + 1/q = 1$  and let  $T_n$  be the linear form defined in  $L^q(K/K_{\mathbb{E}})$  by

$$T_n(\phi) = \int_K f_n(k)\phi(k)dk.$$

Since  $p \leq 2$ , we have  $f_n \in L^p(K/K_{\mathbb{E}})$ . Thus, the linear form  $T_n$  is continuous and

$$|T_n(\phi)| \leq \|f_n\|_p \|\phi\|_q.$$

By Corollary (7), we have  $\|f_n\|_p \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|P_{\lambda_j} f_n\|_{\lambda_j, p}$ .

Hence,

$$|T_n(\phi)| \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|F_n\|_{\lambda_j, p} \|\phi\|_q.$$

Now from

$$\|F_n^t\|_p \leq \|\chi\|_1 \|F^t\|_p = \|F^t\|_p,$$

we deduce that  $\sum_{j=1}^n \|F_n\|_{\lambda_j, p} \leq \sum_{j=1}^n \|F\|_{\lambda_j, p}$  and this implies clearly that

$$|T_n(\phi)| \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|F\|_{\lambda_j, p} \|\phi\|_q.$$

Thus, the linear forms  $T_n$  are uniformly bounded operators in  $L^q(K/K_{\mathbb{E}})$ , with

$$\sup_n \|T_n\| \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|F\|_{\lambda_j, p},$$

where  $\|\cdot\|$  stands for the operator norm.

Next, use the Banach–Alaoglu–Bourbaki theorem to conclude that there exists a subsequence of bounded operators  $(T_{n_j})$  which converges as  $n_j \rightarrow +\infty$  to a bounded operator  $T$  on  $L^q(K/K_{\mathbb{E}})$ , under the  $\star$ -weak topology, with  $\|T\| \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|F\|_{\lambda_j, p}$ . By the Riesz representation theorem, we know that there exists a unique function  $f \in L^p(K/K_{\mathbb{E}})$  such that

$$T(\phi) = \int_K f(k)\phi(k)dk.$$

Now, let

$$\phi_g(k) = \prod_{j=1}^n e^{-(\lambda_j + \eta)\rho_0 H_{\mathbb{E}}(g^{-1}k)}.$$

Then,  $T_n(\phi_g) = F_n(g)$ . Since, on the one hand,

$$\lim_{n \rightarrow +\infty} F_n(g) = F(g)$$

and, on the other hand,

$$\lim_{j \rightarrow +\infty} T_{n_j}(\phi_g) = T(\phi_g),$$

we get  $F(g) = \sum_{j=1}^n P_{\lambda_j} f(g)$ . The estimate  $\|f\|_p \leq \sum_{j=1}^n |c(\lambda_j)|^{-1} \|F\|_{\lambda_j, p}$  follows obviously from the bound of  $T$  and the proof of Theorem (11) is finished.

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